

# Cartan-Eilenberg complexes and Auslander categories

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## Abstract

Let  $R$  be a commutative noetherian ring with a semi-dualizing module  $C$ . The Auslander categories with respect to  $C$  are related through Foxby equivalence:  $\mathcal{A}_C(R) \xrightleftharpoons[\mathbf{R}\mathrm{Hom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} \mathcal{B}_C(R)$ . We firstly intend to extend the Foxby equivalence to Cartan-Eilenberg complexes. To this end, C-E Auslander categories, C-E  $\mathcal{W}$  complexes and C-E  $\mathcal{W}$ -Gorenstein complexes are introduced, where  $\mathcal{W}$  denotes a self-orthogonal class of  $R$ -modules. Moreover, criteria for finiteness of C-E Gorenstein dimensions of complexes in terms of resolution-free characterizations are considered.

*Key Words:* Cartan-Eilenberg complexes; Auslander class; Bass class; Self-orthogonal class; Semi-dualizing module.

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## 1. Introduction

In [20] Verdier introduced the notions of a Cartan-Eilenberg projective (injective) complex and a Cartan-Eilenberg projective (injective) resolution, which originated from [2]. A complex  $P$  is said to be Cartan-Eilenberg projective provided that  $P$ ,  $Z(P)$ ,  $B(P)$  and  $H(P)$  are all complexes of projective modules. Recently, Enochs showed in [6] that Cartan-Eilenberg resolutions can be defined in terms of precovers and preenvelopes by Cartan-Eilenberg projective and injective complexes, and he further introduced Cartan-Eilenberg Gorenstein projective and injective complexes.

Let  $R$  be a commutative noetherian ring. Recall that a semi-dualizing module over  $R$ , which provides a generalization of  $R$  and a dualizing module, is a finite generated module  $C$  such that the homothety morphism  $R \rightarrow \mathbf{R}\mathrm{Hom}_R(C, C)$  is invertible in the derived category  $\mathcal{D}(R)$ . For a given semi-dualizing module  $C$ , consider two triangulated subcategories of the bounded derived category  $\mathcal{D}_b(R)$ ,

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namely the Auslander class  $\mathcal{A}_C(R)$  and the Bass class  $\mathcal{B}_C(R)$ . The adjoint pair of functors  $(C \otimes_R^{\mathbf{L}} -, \mathbf{RHom}_R(C, -))$  on  $\mathcal{D}(R)$  restricts to a pair of equivalences of categories  $\mathcal{A}_C(R) \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} \mathcal{B}_C(R)$  known as Foxby equivalence.

Using [4, Proposition 4.4], it is direct to obtain a commutative diagram:

$$\begin{array}{ccc}
\overline{\mathcal{P}}(R) & \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \overline{\mathcal{P}}_C(R) \\
\downarrow & & \downarrow \\
\mathcal{A}_C(R) & \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \mathcal{B}_C(R) \\
\uparrow & & \uparrow \\
\overline{\mathcal{I}}_C(R) & \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \overline{\mathcal{I}}(R)
\end{array}$$

By  $\overline{\mathcal{P}}(R)$  (resp.  $\overline{\mathcal{P}}_C(R)$ ) we denote the category consisting of complex  $X$  such that  $X \simeq U$  in  $\mathcal{D}_b(R)$ , where  $U$  is a bounded complex of projective (resp.  $C$ -projective) modules; and similarly  $\overline{\mathcal{I}}(R)$  and  $\overline{\mathcal{I}}_C(R)$  are defined.

So it is natural to ask: whether there exist subcategories of  $\mathcal{D}_b(R)$  with respect to Cartan-Eilenberg (C-E for short) complexes, for which one has the related Foxby equivalence? In section 4, we introduce C-E Auslander class  $\text{CE-}\mathcal{A}_C(R)$  and C-E Bass class  $\text{CE-}\mathcal{B}_C(R)$ , and obtain the following diagram by collecting separating results for proving it, which extends the existed Foxby equivalence. We remark that the dual of the diagram exists as well.

**Theorem A.** *Let  $R$  be a commutative noetherian ring. If  $C$  is a semi-dualizing module for  $R$ , then there is a commutative diagram, where the vertical inclusions are full embedding, and the horizontal arrows are equivalences of categories:*

$$\begin{array}{ccc}
\text{CE-}\overline{\mathcal{P}}(R) & \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \text{CE-}\overline{\mathcal{P}}_C(R) \\
\downarrow & & \downarrow \\
\text{CE-}\overline{\mathcal{G}}(\overline{\mathcal{P}})(R) \cap \text{CE-}\mathcal{A}_C(R) & \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \text{CE-}\overline{\mathcal{G}}(\overline{\mathcal{P}}_C)(R) \\
\downarrow & & \downarrow \\
\text{CE-}\mathcal{A}_C(R) & \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \text{CE-}\mathcal{B}_C(R) \\
\downarrow & & \downarrow \\
\mathcal{A}_C(R) & \xrightleftharpoons[\mathbf{RHom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \mathcal{B}_C(R)
\end{array}$$

Since the complexes are related to the classes of projective, injective,  $C$ -projective and  $C$ -injective modules, which are known to be self-orthogonal with respect to  $\text{Ext}$ , in section 2 we generally define and study C-E  $\mathcal{W}$  complexes relative to a self-orthogonal class  $\mathcal{W}$  of  $R$ -modules. In section 3, we focus on studying C-E  $\mathcal{W}$ -Gorenstein complexes by showing that two potential choices for defining these complexes are equivalent.

**Theorem B.** *For a complex  $G$ , the following are equivalent:*

- (1)  *$G$  is a C-E  $\mathcal{W}$ -Gorenstein complex, i.e.  $G$ ,  $B(G)$ ,  $Z(G)$  and  $H(G)$  are complexes of  $\mathcal{W}$ -Gorenstein modules.*
- (2)  *$G$  admits a C-E complete  $\mathcal{W}$  resolution (see Definition 3.4).*

The notions of a dualizing complex and a dualizing module have been extensively developed and applied in many areas such as commutative algebra and algebraic geometry. Recall that a semi-dualizing module  $C$  is dualizing provided that  $C$  has finite injective dimension. Auslander categories with respect to semi-dualizing and dualizing modules (complexes) are used to find criteria for finiteness of Gorenstein homological dimensions of modules and complexes in terms of resolution-free characterizations, see for examples [3, 4, 5, 13, 18]. Finally, homological dimensions with respect to C-E complexes are considered in section 5, and we obtain the following. Moreover, it is worth noting that the complexes considered in Theorem A are precisely those with finite C-E homological dimensions.

**Theorem C.** *Let  $R$  be a commutative noetherian ring with a dualizing module  $C$ ,  $X \in \mathcal{D}_{\square}(R)$ . Assume that  $X \simeq G$  for a C-E Gorenstein projective complex  $G$ . Then the following are equivalent:*

- (1)  *$\text{CE-Gpd}_R X$  is finite.*
- (2)  *$X \in \text{CE-}\mathcal{A}_C(R)$ .*
- (3)  *$\text{Gpd}_R X$ ,  $\text{Gpd}_R Z(X)$ ,  $\text{Gpd}_R B(X)$  and  $\text{Gpd}_R H(X)$  are all finite.*
- (4)  *$X$ ,  $Z(X)$ ,  $B(X)$  and  $H(X)$  are in  $\mathcal{A}_C(R)$ .*

**Notations.** Throughout,  $R$  is a commutative ring. An  $R$ -complex  $X = \cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots$  will be denoted  $(X_i, d_i)$  or  $X$ . The  $n$ th cycle module is defined as  $\text{Ker} d_n$  and is denoted by  $Z_n(X)$ ,  $n$ th boundary module is  $\text{Im} d_{n+1}$  and is denoted by  $B_n(X)$ , and  $n$ th homology module is  $H_n(X) = Z_n(X)/B_n(X)$ . The complexes of cycles and boundaries, and the homology complex of  $X$  is denoted by  $Z(X)$ ,  $B(X)$  and  $H(X)$  respectively. For a given  $R$ -module  $M$ , we let  $S^n(M)$  denote the complex with all entries 0 except  $M$  in degree  $n$ . We let  $D^n(M)$  denote

the complex  $X$  with  $X_n = X_{n-1} = M$  and all other entries 0, and with all maps 0 except  $d_n = 1_M$ .

We use  $\mathcal{C}(R)$  to denote the category of  $R$ -complexes. The symbol “ $\simeq$ ” is used to designate quasi-isomorphisms in the category  $\mathcal{C}(R)$  and isomorphisms in the derived category  $\mathcal{D}(R)$ . The left derived functor of the tensor product functor of  $R$ -complexes is denoted by  $-\otimes_R^{\mathbf{L}}-$ , and  $\mathbf{R}\mathrm{Hom}_R(-, -)$  denotes the right derived functor of the homomorphism functor of complexes.

Let  $C$  be a semi-dualizing module over a noetherian ring  $R$ . The Auslander categories with respect to  $C$ , denoted by  $\mathcal{A}_C(R)$  and  $\mathcal{B}_C(R)$ , are the full subcategories of  $\mathcal{D}_b(R)$  whose objects are specified as follows:

$$\mathcal{A}_C(R) = \left\{ X \in \mathcal{D}_b(R) \left| \begin{array}{l} \eta_X : X \rightarrow \mathbf{R}\mathrm{Hom}_R(C, C \otimes_R^{\mathbf{L}} X) \text{ is an iso-} \\ \text{morphism in } \mathcal{D}(R), \text{ and } C \otimes_R^{\mathbf{L}} X \in \mathcal{D}_b(R) \end{array} \right. \right\}$$

and

$$\mathcal{B}_C(R) = \left\{ Y \in \mathcal{D}_b(R) \left| \begin{array}{l} \varepsilon_Y : C \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}(C, Y) \rightarrow Y \text{ is an isomor-} \\ \text{phism in } \mathcal{D}(R), \text{ and } \mathbf{R}\mathrm{Hom}(C, Y) \in \mathcal{D}_b(R) \end{array} \right. \right\},$$

where  $\eta$  and  $\varepsilon$  denote the unit and counit of the adjoint pair  $(C \otimes_R^{\mathbf{L}} -, \mathbf{R}\mathrm{Hom}(C, -))$ .

## 2. Cartan-Eilenberg $\mathcal{W}$ complexes

Let  $\mathcal{W}$  be a class of  $R$ -modules.  $\mathcal{W}$  is called self-orthogonal if it satisfies the following condition:

$$\mathrm{Ext}_R^i(W, W') = 0 \text{ for all } W, W' \in \mathcal{W} \text{ and all } i \geq 1.$$

In the following,  $\mathcal{W}$  always denotes a self-orthogonal class of  $R$ -modules which is closed under extensions, finite direct sums and direct summands. Geng and Ding enumerated in [10, Remark 2.3] some interesting examples of self-orthogonal classes.

**Definition 2.1.** *A complex  $X$  is said to be a C-E  $\mathcal{W}$  complex if  $X$ ,  $Z(X)$ ,  $B(X)$  and  $H(X)$  are complexes each of whose terms belongs to  $\mathcal{W}$ .*

**Remark 2.2.** (1) *For any module  $M \in \mathcal{W}$  and any  $n \in \mathbb{Z}$ ,  $S^n(M)$  and  $D^n(M)$  are C-E  $\mathcal{W}$  complexes.*

(2) *In particular, if  $\mathcal{W}$  denotes the class of projective (resp. injective) modules, then C-E  $\mathcal{W}$  complexes are precisely C-E projective (resp. C-E injective) complexes.*

We will frequently consider complexes  $X$  with  $d^X = 0$ . Such a complex is completely determined by its family of terms  $(X_n)_{n \in \mathbb{Z}}$ , i.e. by the underlying structure of  $X$  as a graded module with the grading over  $\mathbb{Z}$ . So by the term graded module we will mean a complex  $X$  with  $d^X = 0$ .

Recall that  $X$  is called a  $\mathcal{W}$  complex [15] if  $X$  is exact and  $Z_n(X) \in \mathcal{W}$  for any  $n \in \mathbb{Z}$ . We will denote the class of  $\mathcal{W}$  complexes by  $\widetilde{\mathcal{W}}$ .

**Proposition 2.3.**  *$X$  is a C-E  $\mathcal{W}$  complex if and only if  $X$  can be divided into direct sums  $X = X' \oplus X''$  where  $X' \in \widetilde{\mathcal{W}}$  and  $X''$  is a graded module with all items in  $\mathcal{W}$ .*

*Proof.* Since  $X' \in \widetilde{\mathcal{W}}$  is exact,  $B_n(X') = Z_n(X') \in \mathcal{W}$ ,  $H_n(X') = 0$  for all  $n \in \mathbb{Z}$ . Then  $X'$  is a C-E  $\mathcal{W}$  complex. It is easy to see  $X''$  is a C-E  $\mathcal{W}$  complex. Then such direct sum is a C-E  $\mathcal{W}$  complex.

Conversely, suppose that  $X$  is a C-E  $\mathcal{W}$  complex. We have the exact sequences of  $R$ -modules  $0 \rightarrow B_n(X) \rightarrow Z_n(X) \rightarrow H_n(X) \rightarrow 0$  and  $0 \rightarrow Z_n(X) \rightarrow X_n \rightarrow B_{n-1}(X) \rightarrow 0$ . Since  $Z_n(X), B_n(X), H_n(X) \in \mathcal{W}$  for all  $n \in \mathbb{Z}$ , each sequence splits. This allows us to write  $X_n = B_n(X) \oplus H_n(X) \oplus B_{n-1}(X)$ . Then

$$d_n : X_n = B_n(X) \oplus H_n(X) \oplus B_{n-1}(X) \rightarrow X_{n-1} = B_{n-1}(X) \oplus H_{n-1}(X) \oplus B_{n-2}(X)$$

is the map  $(x, y, z) \rightarrow (z, 0, 0)$ . Let  $X' = \bigoplus_{n \in \mathbb{Z}} D^n(B_{n-1}(X))$ ,  $X'' = \bigoplus_{n \in \mathbb{Z}} S^n(H_n(X))$ . We then obtain the desired direct sum decomposition  $X = X' \oplus X''$ .  $\square$

**Corollary 2.4.** ([6, Proposition 3.4]) *A complex  $X$  is C-E projective if and only if  $X$  can be divided into direct sums  $X = X' \oplus X''$ , where  $X'$  is a projective complex and  $X''$  is a graded module with all items being projective.*

Let  $R$  be a noetherian ring with a semi-dualizing module  $C$ . An  $R$ -module is  $C$ -projective if it has the form  $C \otimes_R P$  for some projective  $R$ -module  $P$ . An  $R$ -module is  $C$ -injective if it has the form  $\text{Hom}_R(C, E)$  for some injective  $R$ -module  $E$ . Let  $\mathcal{P}_C = \{C \otimes_R P \mid P \text{ is a projective } R\text{-module}\}$  and  $\mathcal{I}_C = \{\text{Hom}_R(C, E) \mid E \text{ is an injective } R\text{-module}\}$  denote the class of  $C$ -projective and  $C$ -injective modules, respectively.

By [10, Theorem 3.1],  $\mathcal{P}_C = \text{Add}C$  and  $\mathcal{I}_C = \text{Prod}C^+$ , where  $\text{Add}C$  stands for the category consisting of all modules isomorphic to direct summands of direct sums of copies of  $C$ , and  $\text{Prod}C^+$  the category consisting of all modules isomorphic to direct summands of direct products of copies of  $C^+ = \text{Hom}_R(C, Q)$  with  $Q$  an injective cogenerator. Thus  $\mathcal{P}_C$  and  $\mathcal{I}_C$  are self-orthogonal. If we put  $\mathcal{W} = \mathcal{P}_C$  (resp.  $\mathcal{W} = \mathcal{I}_C$ ), then a C-E  $\mathcal{W}$  complex above is particularly called a C-E  $C$ -projective (resp. C-E  $C$ -injective) complex.

**Corollary 2.5.** *Let  $R$  be a noetherian ring with a semi-dualizing module  $C$  and  $X$  an  $R$ -complex. Then  $X$  is C-E  $C$ -projective if and only if  $X$  can be divided into direct sums  $X = X' \oplus X''$  where  $X' \in \widetilde{\mathcal{P}_C}$  and  $X''$  is a graded module with all items in  $\mathcal{P}_C$ .*

Similarly,  $X$  is C-E  $C$ -injective if and only if  $X$  can be divided into direct sums  $X = X' \oplus X''$  where  $X' \in \widetilde{\mathcal{I}}_C$  and  $X''$  is a graded module with all items in  $\mathcal{I}_C$ .

### 3. Cartan-Eilenberg $\mathcal{W}$ -Gorenstein complexes

Recall that an  $R$ -module  $M$  is said to be  $\mathcal{W}$ -Gorenstein [10, Definition 2.2] if there exists an exact sequence

$$W_\bullet = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W_{-1} \longrightarrow W_{-2} \longrightarrow \cdots$$

of modules in  $\mathcal{W}$  such that  $M = \text{Ker}(W_{-1} \rightarrow W_{-2})$  and  $W_\bullet$  is  $\text{Hom}_R(\mathcal{W}, -)$  and  $\text{Hom}_R(-, \mathcal{W})$  exact. In this case,  $W_\bullet$  is called a complete  $\mathcal{W}$ -resolution of  $M$ . This covers a various of examples by different choices of  $\mathcal{W}$ , for instance, Gorenstein projective and Gorenstein injective modules.

We note that the class of  $\mathcal{W}$ -Gorenstein modules is just the class  $\mathcal{G}(\mathcal{W})$  introduced by Sather-Wagstaff and coauthors [17] when the abelian category  $\mathcal{A}$  is taken to be the category of  $R$ -modules. So  $\mathcal{G}(\mathcal{W}) = \mathcal{G}^2(\mathcal{W})$ . The symbol  $\mathcal{G}^2(\mathcal{W})$  denotes the class of  $R$ -module  $M$  defined by an iteration of the procedure used to define  $\mathcal{W}$ -Gorenstein modules, that is, there exists an exact sequence

$$G_\bullet = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G_{-1} \longrightarrow G_{-2} \longrightarrow \cdots$$

of  $\mathcal{W}$ -Gorenstein modules such that  $M = \text{Ker}(G_{-1} \rightarrow G_{-2})$  and  $G_\bullet$  remains exact by applying  $\text{Hom}_R(G, -)$  and  $\text{Hom}_R(-, G)$  for any  $\mathcal{W}$ -Gorenstein module  $G$ .

**Definition 3.1.** *A complex  $X$  is said to be a C-E  $\mathcal{W}$ -Gorenstein complex if  $X$ ,  $Z(X)$ ,  $B(X)$  and  $H(X)$  are complexes consisting of  $\mathcal{W}$ -Gorenstein modules.*

We will show that one can also use a modification of the definition of  $\mathcal{W}$ -Gorenstein module to define such a complex. First, we need to recall the following definition.

**Definition 3.2.** ([6, Definition 5.3]) *A complex of complexes*

$$\mathcal{C} = \cdots \longrightarrow C^2 \longrightarrow C^1 \longrightarrow C^0 \longrightarrow C^{-1} \longrightarrow \cdots$$

*is said to be C-E exact if the following sequences are all exact:*

- (1)  $\cdots \longrightarrow C^1 \longrightarrow C^0 \longrightarrow C^{-1} \longrightarrow \cdots$ .
- (2)  $\cdots \longrightarrow Z(C^1) \longrightarrow Z(C^0) \longrightarrow Z(C^{-1}) \longrightarrow \cdots$ .
- (3)  $\cdots \longrightarrow B(C^1) \longrightarrow B(C^0) \longrightarrow B(C^{-1}) \longrightarrow \cdots$ .
- (4)  $\cdots \longrightarrow C^1/Z(C^1) \longrightarrow C^0/Z(C^0) \longrightarrow C^{-1}/Z(C^{-1}) \longrightarrow \cdots$ .
- (5)  $\cdots \longrightarrow C^1/B(C^1) \longrightarrow C^0/B(C^0) \longrightarrow C^{-1}/B(C^{-1}) \longrightarrow \cdots$ .
- (6)  $\cdots \longrightarrow H(C^1) \longrightarrow H(C^0) \longrightarrow H(C^{-1}) \longrightarrow \cdots$ .

**Remark 3.3.** *In the above definition, exactness of (1) and (2) implies exactness of all (1)-(6), and exactness of (1) and (5) implies exactness of all (1)-(6).*

In the following, we focus on C-E  $\mathcal{W}$ -Gorenstein complexes and we show that such complexes can be obtained by a so-called C-E complete  $\mathcal{W}$ -resolution.

**Definition 3.4.** *For a complex  $G$ , by a C-E complete  $\mathcal{W}$ -resolution of  $G$  we mean a C-E exact sequence*

$$\mathbb{W} = \cdots \longrightarrow W^1 \longrightarrow W^0 \longrightarrow W^{-1} \longrightarrow W^{-2} \longrightarrow \cdots$$

*of C-E  $\mathcal{W}$  complexes with  $G = \text{Ker}(W^{-1} \rightarrow W^{-2})$ , such that it remains exact after applying  $\text{Hom}_{\mathcal{C}(R)}(V, -)$  and  $\text{Hom}_{\mathcal{C}(R)}(-, V)$  for any C-E  $\mathcal{W}$  complex  $V$ .*

There are a few useful adjoint relationships between the category of  $R$ -modules and the category of  $R$ -complexes.

**Lemma 3.5.** ([11, Lemma 3.1]) *For any  $R$ -module  $M$  and any  $R$ -complex  $X$ , we have the following natural isomorphisms:*

- (1)  $\text{Hom}_{\mathcal{C}(R)}(D^n(M), X) \cong \text{Hom}_R(M, X_n)$ .
- (2)  $\text{Hom}_{\mathcal{C}(R)}(S^n(M), X) \cong \text{Hom}_R(M, Z_n(X))$ .
- (3)  $\text{Hom}_{\mathcal{C}(R)}(X, D^n(M)) \cong \text{Hom}_R(X_{n-1}, M)$ .
- (4)  $\text{Hom}_{\mathcal{C}(R)}(X, S^n(M)) \cong \text{Hom}_R(X_n/B_n(X), M)$ .

Now we are in a position to prove the following result in Theorem B from the introduction. In the next, we use both the subscript notation for degrees of complex and the superscript notation to distinguish complexes: for example, if  $(C^i)_{i \in \mathbb{Z}}$  is a family of complexes, then  $C_n^i$  denotes the degree- $n$  term of the complex  $C^i$ .

**Theorem 3.6.** *For a complex  $G$ , the following are equivalent:*

- (1)  $G$  is a C-E  $\mathcal{W}$ -Gorenstein complex.
- (2)  $G$  admits a C-E complete  $\mathcal{W}$  resolution.
- (3)  $B(G)$  and  $H(G)$  are complexes consisting of  $\mathcal{W}$ -Gorenstein modules.

*Proof.* (1) $\implies$ (2) For any  $n \in \mathbb{Z}$ , consider the exact sequence of modules  $0 \rightarrow B_n(G) \rightarrow Z_n(G) \rightarrow H_n(G) \rightarrow 0$ , where  $B_n(G)$  and  $H_n(G)$  are  $\mathcal{W}$ -Gorenstein. Suppose  $W^{B_n(G)}$  and  $W^{H_n(G)}$  are complete  $\mathcal{W}$ -resolutions of  $B_n(G)$  and  $H_n(G)$ , respectively. By the Horseshoe Lemma, we can construct a complete  $\mathcal{W}$  resolution of  $Z_n(G)$ :  $W^{Z_n(G)} = W^{B_n(G)} \oplus W^{H_n(G)}$ . Similarly, consider the exact sequence of modules  $0 \rightarrow Z_n(G) \rightarrow G_n \rightarrow B_{n-1}(G) \rightarrow 0$ , and we can construct a complete  $\mathcal{W}$ -resolution of  $G_n$ :

$$W^{G_n} = W^{Z_n(G)} \oplus W^{B_{n-1}(G)} = W^{B_n(G)} \oplus W^{H_n(G)} \oplus W^{B_{n-1}(G)}.$$

Set  $W_n^i = W_i^{B_n(G)} \oplus W_i^{H_n(G)} \oplus W_i^{B_{n-1}(G)}$  and  $d_n^{W^i} : W_n^i \rightarrow W_{n-1}^i$  which maps  $(x, y, z)$  to  $(z, 0, 0)$  for all  $i, n \in \mathbb{Z}$ . Then  $(W^i, d^{W^i})$  is a complex such that  $G_n = \text{Ker}(W_n^{-1} \rightarrow W_n^{-2})$ .

Consider the complex of complexes

$$\mathbb{W} = \cdots \longrightarrow W^1 \longrightarrow W^0 \longrightarrow W^{-1} \longrightarrow W^{-2} \longrightarrow \cdots$$

For any  $n \in \mathbb{Z}$ ,  $\mathbb{W}_n = \cdots \rightarrow W_n^1 \rightarrow W_n^0 \rightarrow W_n^{-1} \rightarrow \cdots$  is a complete  $\mathcal{W}$ -resolution of  $G_n$ , and  $Z_n(\mathbb{W}) = \cdots \rightarrow Z_n(W^1) \rightarrow Z_n(W^0) \rightarrow Z_n(W^{-1}) \rightarrow \cdots$  is a complete  $\mathcal{W}$ -resolution of  $Z_n(G)$ , so they both are exact. Hence, we can get that  $\mathbb{W}$  is C-E exact. It is easily seen that  $W^i$  is a C-E  $\mathcal{W}$  complex for all  $i \in \mathbb{Z}$  and  $G = \text{Ker}(W^{-1} \rightarrow W^{-2})$ . It remains to prove that  $\mathbb{W}$  is still exact when  $\text{Hom}_{\mathcal{C}(R)}(V, -)$  and  $\text{Hom}_{\mathcal{C}(R)}(-, V)$  applied to it for any C-E  $\mathcal{W}$  complex  $V$ .

However, it suffices to prove, by Proposition 2.3, that the assertion holds when we pick  $V$  particularly as  $V = D^n(M)$  and  $V = S^n(M)$  for any module  $M \in \mathcal{W}$  and all  $n \in \mathbb{Z}$ . By Lemma 3.5, there is a natural isomorphism  $\text{Hom}_{\mathcal{C}(R)}(D^n(M), \mathbb{W}) \cong \text{Hom}_R(M, \mathbb{W}_n)$ , i.e. there is a commutative diagram

$$\begin{array}{ccccccc} \cdots & \rightarrow & \text{Hom}_{\mathcal{C}(R)}(D^n(M), W^1) & \rightarrow & \text{Hom}_{\mathcal{C}(R)}(D^n(M), W^0) & \rightarrow & \text{Hom}_{\mathcal{C}(R)}(D^n(M), W^{-1}) \rightarrow \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \cdots & \longrightarrow & \text{Hom}_R(M, W_n^1) & \longrightarrow & \text{Hom}_R(M, W_n^0) & \longrightarrow & \text{Hom}_R(M, W_n^{-1}) \longrightarrow \cdots \end{array}$$

Note that  $\mathbb{W}_n$  is a complete  $\mathcal{W}$ -resolutions of  $G_n$ , then the exactness of the upper row follows since the bottom row is exact. Similarly, we have exactness of  $\text{Hom}_{\mathcal{C}(R)}(\mathbb{W}, D^n(M))$  by  $\text{Hom}_{\mathcal{C}(R)}(\mathbb{W}, D^n(M)) \cong \text{Hom}_R(\mathbb{W}_{n-1}, M)$ , and exactness of  $\text{Hom}_{\mathcal{C}(R)}(S^n(M), \mathbb{W})$  by  $\text{Hom}_{\mathcal{C}(R)}(S^n(M), \mathbb{W}) \cong \text{Hom}_R(M, Z_n(\mathbb{W}))$ .

Moreover, it yields from the exact sequence  $0 \rightarrow H_n(W^i) \rightarrow W_n^i/B_n(W^i) \rightarrow B_{n-1}(W^i) \rightarrow 0$  that  $W_n^i/B_n(W^i) \in \mathcal{W}$ , and then there exists an exact sequence of complexes

$$0 \longrightarrow B_n(\mathbb{W}) \longrightarrow \mathbb{W}_n \longrightarrow \mathbb{W}_n/B_n(\mathbb{W}) \longrightarrow 0,$$

which is split degreeewise. Noting that  $B_n(\mathbb{W})$  is a complete  $\mathcal{W}$ -resolution of  $B_n(G)$ , the complex  $\text{Hom}_R(B_n(\mathbb{W}), M)$  is exact. From the exact sequence of  $\mathbb{Z}$ -complexes

$$0 \longrightarrow \text{Hom}_R(\mathbb{W}_n/B_n(\mathbb{W}), M) \longrightarrow \text{Hom}_R(\mathbb{W}_n, M) \longrightarrow \text{Hom}_R(B_n(\mathbb{W}), M) \longrightarrow 0,$$

it yields that  $\text{Hom}_R(\mathbb{W}_n/B_n(\mathbb{W}), M)$  is exact since the other two items are so. Hence,  $\text{Hom}_{\mathcal{C}(R)}(\mathbb{W}, S^n(M)) \cong \text{Hom}_R(\mathbb{W}_n/B_n(\mathbb{W}), M)$  is exact.

(2) $\implies$  (1) Suppose that

$$\mathbb{W} = \cdots \longrightarrow W^1 \longrightarrow W^0 \longrightarrow W^{-1} \longrightarrow W^{-2} \longrightarrow \cdots$$

is a C-E complete  $\mathcal{W}$ -resolution such that  $G = \text{Ker}(W^{-1} \rightarrow W^{-2})$ . Let  $M \in \mathcal{W}$ . It follows by the exactness of  $\text{Hom}_{\mathcal{C}(R)}(D^n(M), \mathbb{W})$  and  $\text{Hom}_{\mathcal{C}(R)}(\mathbb{W}, D^{n+1}(M))$  that

$$\mathbb{W}_n = \cdots \longrightarrow W_n^1 \longrightarrow W_n^0 \longrightarrow W_n^{-1} \longrightarrow W_n^{-2} \longrightarrow \cdots$$



remains exact after applying  $\text{Hom}_R(M, -)$  and  $\text{Hom}_R(-, M)$ . Then,  $\mathbb{W}_n$  is a complete  $\mathcal{W}$ -resolution of  $G_n = \text{Ker}(W_n^{-1} \rightarrow W_n^{-2})$ , and thus  $G$  is a complex of  $\mathcal{W}$ -Gorenstein modules.

We now argue that  $Z(G)$  is also a complex of  $\mathcal{W}$ -Gorenstein modules. So we must show that  $Z_n(G)$  is  $\mathcal{W}$ -Gorenstein for any  $n$ . We have a candidate for a complete  $\mathcal{W}$ -resolution of  $Z_n(G)$ , namely an exact sequence

$$Z_n(\mathbb{W}) = \cdots \longrightarrow Z_n(W^1) \longrightarrow Z_n(W^0) \longrightarrow Z_n(W^{-1}) \longrightarrow \cdots$$

with each module  $Z_n(W^i) \in \mathcal{W}$ , such that  $Z_n(G) = \text{Ker}(Z_n(W^{-1}) \rightarrow Z_n(W^{-2}))$ . Clearly,  $\text{Hom}_R(M, Z_n(\mathbb{W})) \cong \text{Hom}_{\mathcal{C}(R)}(S^n(M), \mathbb{W})$  is exact. Moreover, in the exact sequence  $0 \rightarrow \text{Hom}_R(\mathbb{W}_n/B_n(\mathbb{W}), M) \rightarrow \text{Hom}_R(\mathbb{W}_n, M) \rightarrow \text{Hom}_R(B_n(\mathbb{W}), M) \rightarrow 0$  of complexes,  $\text{Hom}_R(\mathbb{W}_n, M)$  and  $\text{Hom}_R(\mathbb{W}_n/B_n(\mathbb{W}), M) \cong \text{Hom}_{\mathcal{C}(R)}(\mathbb{W}, S^n(M))$  are exact, then so is  $\text{Hom}_R(B_n(\mathbb{W}), M)$ . Now consider the following exact sequence of complexes

$$0 \longrightarrow Z_n(\mathbb{W}) \longrightarrow \mathbb{W}_n \longrightarrow B_{n-1}(\mathbb{W}) \longrightarrow 0,$$

which is split degreewise and yields an exact sequence of  $\mathbb{Z}$ -complexes

$$0 \longrightarrow \text{Hom}_R(B_{n-1}(\mathbb{W}), M) \longrightarrow \text{Hom}_R(\mathbb{W}_n, M) \longrightarrow \text{Hom}_R(Z_n(\mathbb{W}), M) \longrightarrow 0.$$

Then  $\text{Hom}_R(Z_n(\mathbb{W}), M)$  is exact. This implies that  $Z_n(G)$  is  $\mathcal{W}$ -Gorenstein.

Furthermore, we have from the exact sequence  $0 \rightarrow Z_{n+1}(\mathbb{W}) \rightarrow \mathbb{W}_{n+1} \rightarrow B_n(\mathbb{W}) \rightarrow 0$  that  $B_n(\mathbb{W})$  is a complete  $\mathcal{W}$ -resolution of  $B_n(G)$ , and then from  $0 \rightarrow B_n(\mathbb{W}) \rightarrow Z_n(\mathbb{W}) \rightarrow H_n(\mathbb{W}) \rightarrow 0$  that  $H_n(\mathbb{W})$  is a complete  $\mathcal{W}$ -resolution of  $H_n(G)$ . Hence,  $B(G)$  and  $H(G)$  are both complexes of  $\mathcal{W}$ -Gorenstein modules.

(1) $\implies$ (3) is trivial.

(3) $\implies$ (1) Since the class  $\mathcal{G}(\mathcal{W})$  of  $\mathcal{W}$ -Gorenstein modules is closed under extensions [17, Corollary 4.5], the assertion follows from the exact sequences  $0 \rightarrow B_n(G) \rightarrow Z_n(G) \rightarrow H_n(G) \rightarrow 0$  and  $0 \rightarrow Z_n(G) \rightarrow G_n \rightarrow B_{n-1}(G) \rightarrow 0$ .  $\square$

It is an important question to establish relationships between a complex  $X$  and the modules  $X_n, n \in \mathbb{Z}$ . If  $R$  is an  $n$ -Gorenstein ring, Enochs and Garcia Rozas showed in [8] that a complex  $X$  is Gorenstein projective (resp. Gorenstein injective) if and only if  $X_n$  is a Gorenstein projective (resp. Gorenstein injective) module for any  $n \in \mathbb{Z}$ . This has been further developed by Liu and Zhang [16] and Yang [22], and now we know that the same result holds over any associative ring.

In [15, Section 5.1], the author introduced the notion of  $\widetilde{\mathcal{W}}$ -Gorenstein complex, analogous to the definition of  $\mathcal{W}$ -Gorenstein module, by replacing the modules in  $\mathcal{W}$  with complexes in  $\widetilde{\mathcal{W}}$ . It is proved that ([15, Sect. 5.1, Theorem A]): a complex  $X$  is  $\widetilde{\mathcal{W}}$ -Gorenstein if and only if  $X_n$  is a  $\mathcal{W}$ -Gorenstein module for each  $n \in \mathbb{Z}$ .

**Corollary 3.7.** *For a complex  $G$ , the following are equivalent:*

- (1)  $G$  is a C-E  $\mathcal{W}$ -Gorenstein complex.
- (2)  $G$ ,  $Z(G)$ ,  $B(G)$  and  $H(G)$  are  $\widetilde{\mathcal{W}}$ -Gorenstein complexes.
- (3)  $B(G)$  and  $H(G)$  are  $\widetilde{\mathcal{W}}$ -Gorenstein complexes.

In particular, if we set  $\mathcal{W}$  to be the class of injective modules  $\mathcal{I}$ , then

**Corollary 3.8.** ([6, Theorem 8.5]) *For a complex  $G$ , the following are equivalent:*

- (1)  $G$  has a C-E complete injective resolution.
- (2)  $G$ ,  $Z(G)$ ,  $B(G)$  and  $H(G)$  are complexes consisting of Gorenstein injective modules.

We will not state here, but there are dual results about C-E Gorenstein projective complexes if  $\mathcal{W}$  is the class of projective modules  $\mathcal{P}$ . In particular, set  $\mathcal{W} = \mathcal{P}_C$  and  $\mathcal{W} = \mathcal{I}_C$  respectively, where  $C$  is a given semi-dualizing module over a commutative noetherian ring  $R$ . In [10],  $\mathcal{P}_C$ -Gorenstein and  $\mathcal{I}_C$ -Gorenstein modules are named respectively  $C$ -Gorenstein projective and  $C$ -Gorenstein injective modules. Accordingly, C-E  $\mathcal{P}_C$ -Gorenstein and C-E  $\mathcal{I}_C$ -Gorenstein complexes are called C-E  $C$ -Gorenstein projective and C-E  $C$ -Gorenstein injective complexes respectively.

#### 4. Foxby equivalence

This section is devoted to prove Theorem A from the introduction. The proof is divided into the following results.

Throughout this section,  $R$  is a noetherian ring, and  $C$  is a given semi-dualizing module over  $R$ . Foxby [9] studied modules in Auslander class  $\mathcal{A}_C^0(R)$  and Bass class  $\mathcal{B}_C^0(R)$ , where

$$\mathcal{A}_C^0(R) = \left\{ M \in R\text{-Mod} \left| \begin{array}{l} \text{Tor}_i^R(C, M) = 0 = \text{Ext}_R^i(C, C \otimes_R M), \text{ and the} \\ \text{map } M \longrightarrow \text{Hom}_R(C, C \otimes_R M) \text{ is an isomorphism} \end{array} \right. \right\},$$

and

$$\mathcal{B}_C^0(R) = \left\{ N \in R\text{-Mod} \left| \begin{array}{l} \text{Ext}_R^i(C, N) = 0 = \text{Tor}_i^R(C, \text{Hom}_R(C, N)), \text{ and the} \\ \text{map } C \otimes_R \text{Hom}_R(C, N) \longrightarrow N \text{ is an isomorphism} \end{array} \right. \right\}.$$

We always take  $C$  as a complex concentrated in degree zero, and then it is a semi-dualizing complex in the sense of [4, Definition 2.1]. By [4, Observation 4.10],  $\mathcal{A}_C^0(R)$  and  $\mathcal{B}_C^0(R)$  coincide with the subcategories of  $\mathcal{A}_C(R)$  and  $\mathcal{B}_C(R)$  consisting of  $R$ -complexes concentrated in degree zero, respectively.

**Definition 4.1.** *Let  $C$  be a semi-dualizing module over a noetherian ring  $R$ . The C-E Auslander class and C-E Bass class with respect to  $C$ , denoted by  $\text{CE-}\mathcal{A}_C(R)$*

and  $\text{CE-}\mathcal{B}_C(R)$ , are the full subcategories of  $\mathcal{D}_b(R)$  whose objects are specified as follows:

$$\text{CE-}\mathcal{A}_C(R) = \left\{ X \in \mathcal{D}_b(R) \left| \begin{array}{l} X \simeq A, \text{ where } A, Z(A), B(A) \text{ and } H(A) \text{ are} \\ \text{complexes consisting of modules in } \mathcal{A}_C^0(R) \end{array} \right. \right\}$$

and

$$\text{CE-}\mathcal{B}_C(R) = \left\{ X \in \mathcal{D}_b(R) \left| \begin{array}{l} X \simeq D, \text{ where } D, Z(D), B(D) \text{ and } H(D) \text{ are} \\ \text{complexes consisting of modules in } \mathcal{B}_C^0(R) \end{array} \right. \right\}.$$

**Proposition 4.2.** *Let  $R$  be a noetherian ring with a semi-dualizing module  $C$ ,  $X$  an  $R$ -complex. If  $X \in \text{CE-}\mathcal{A}_C(R)$ , then the complexes  $X$ ,  $Z(X)$ ,  $B(X)$  and  $H(X)$  are all in  $\mathcal{A}_C(R)$ . Dually, if  $X \in \text{CE-}\mathcal{B}_C(R)$ , then the complexes  $X$ ,  $Z(X)$ ,  $B(X)$  and  $H(X)$  are all in  $\mathcal{B}_C(R)$ .*

*Proof.* Let  $X \in \text{CE-}\mathcal{A}_C(R)$ . Then there exists an isomorphism  $X \simeq A$  in  $\mathcal{D}(R)$ , where  $A$  is a complex such that  $A$ ,  $Z(A)$ ,  $B(A)$  and  $H(A)$  are complexes consisting of modules in  $\mathcal{A}_C^0(R)$ . Set  $\sup(X) = s$ ,  $\inf(X) = i$ . Consider a truncated complex

$$A_{(s,i)} = 0 \longrightarrow A_s/B_s(A) \longrightarrow A_{s-1} \longrightarrow \cdots \longrightarrow A_{i+1} \longrightarrow Z_i(A) \longrightarrow 0.$$

By noting that in the exact sequence  $0 \rightarrow B_s(A) \rightarrow A_s \rightarrow A_s/B_s(A) \rightarrow 0$  the first two entries are in  $\mathcal{A}_C^0(R)$ , the third is also in  $\mathcal{A}_C^0(R)$ , and then we have  $A \simeq A_{(s,i)} \in \text{CE-}\mathcal{A}_C(R)$ . Thus, we may choose  $A$  to be a bounded complex consisting of modules in  $\mathcal{A}_C^0(R)$ .

Let  $\alpha : P^\bullet \rightarrow C$  be a projective resolution of the semi-dualizing module  $C$ , where  $P^\bullet = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$ . Then  $\alpha$  is a quasi-isomorphism, and we represent  $C \otimes_R^{\mathbf{L}} X \simeq C \otimes_R^{\mathbf{L}} A$  by the complex  $P^\bullet \otimes_R A$ . For any  $n \in \mathbb{Z}$ , it follows from  $\text{Tor}_R^i(C, A_n) = 0$  that  $\alpha \otimes_R A_n : P^\bullet \otimes_R A_n \rightarrow C \otimes_R A_n$  is a quasi-isomorphism. By [5, Proposition 2.14],  $\alpha \otimes_R A : P^\bullet \otimes_R A \rightarrow C \otimes_R A$  is then a quasi-isomorphism, and hence  $C \otimes_R^{\mathbf{L}} A \simeq P^\bullet \otimes_R A \simeq C \otimes_R A \in \mathcal{D}_b(R)$ .

It follows that  $\text{Hom}_R(C, C \otimes_R A_n) \rightarrow \text{Hom}_R(P^\bullet, C \otimes_R A_n)$  is a quasi-isomorphism since  $\text{Ext}_R^i(C, C \otimes_R A_n) = 0$ . Then, we have a quasi-isomorphism  $\text{Hom}_R(\alpha, C \otimes_R A) : \text{Hom}_R(C, C \otimes_R A) \rightarrow \text{Hom}_R(P^\bullet, C \otimes_R A)$  by [5, Proposition 2.7]. Hence  $\mathbf{R}\text{Hom}_R(C, C \otimes_R^{\mathbf{L}} A) \simeq \text{Hom}_R(P^\bullet, C \otimes_R A) \simeq \text{Hom}_R(C, C \otimes_R A)$ . Moreover,  $A \rightarrow \text{Hom}_R(C, C \otimes_R A)$  is a canonical isomorphism. Then  $A \rightarrow \mathbf{R}\text{Hom}_R(C, C \otimes_R^{\mathbf{L}} A)$  is an isomorphism in  $\mathcal{D}(R)$ , and this implies  $X \simeq A \in \mathcal{A}_C(R)$ . Similarly, the bounded complex  $H(A)$  is also in  $\mathcal{A}_C(R)$ , and so  $H(X) \simeq H(A) \in \mathcal{A}_C(R)$ . In the next, we need to prove that the complexes  $Z(X)$  and  $B(X)$  are in  $\mathcal{A}_C(R)$ .

Conversely, we suppose  $Z(X)$  is not in  $\mathcal{A}_C(R)$ . Note that  $\mathcal{A}_C(R)$  is a triangulated subcategory of  $\mathcal{D}_b(R)$ , which satisfies 2-out-of-3 property, that is, if any two items

in an exact sequence are in  $\mathcal{A}_C(R)$  then so is the third. Then the exact sequence  $0 \rightarrow B(X) \rightarrow Z(X) \rightarrow H(X) \rightarrow 0$  yields that  $B(X) \notin \mathcal{A}_C(R)$  (otherwise, contradict to  $Z(X) \notin \mathcal{A}_C(R)$ ). Moreover, there is an exact sequence  $0 \rightarrow Z(X) \rightarrow X \rightarrow B(X)[1] \rightarrow 0$ , where  $B(X)[1]$  is the complex obtained by shifting  $B(X)$  one-degree to the left, and it yields that  $X \notin \mathcal{A}_C(R)$ . Hence, a contradiction occurs, which implies that both  $Z(X)$  and  $B(X)$  are in  $\mathcal{A}_C(R)$ .

The rest of the assertions can be proved dually, and then is omitted.  $\square$

**Proposition 4.3.** *There is an equivalence of categories:*

$$\text{CE-}\mathcal{A}_C(R) \xrightleftharpoons[\mathbf{R}\text{Hom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} \text{CE-}\mathcal{B}_C(R).$$

*Proof.* Let  $X \in \text{CE-}\mathcal{A}_C(R)$ . Then there exists an isomorphism  $X \simeq A$  in  $\mathcal{D}(R)$ , where  $A$  is a bounded complex such that  $A$ ,  $Z(A)$ ,  $B(A)$  and  $H(A)$  are complexes consisting of modules in  $\mathcal{A}_C^0(R)$ .

It follows from the arguments above that  $C \otimes_R^{\mathbf{L}} X \simeq C \otimes_R A$ . Obviously,  $C \otimes_R A$  is a complex of modules in  $\mathcal{B}_C^0(R)$ . It remains to prove that  $Z(C \otimes_R A)$ ,  $B(C \otimes_R A)$  and  $H(C \otimes_R A)$  are all complexes of modules in  $\mathcal{B}_C^0(R)$ .

For any  $n \in \mathbb{Z}$ , consider the exact sequence  $0 \rightarrow Z_n(A) \rightarrow A_n \xrightarrow{d_n^A} B_{n-1}(A) \rightarrow 0$ . Since  $B_{n-1}(A) \in \mathcal{A}_C^0(R)$ , then there is an exact sequence

$$0 \longrightarrow C \otimes_R Z_n(A) \longrightarrow C \otimes_R A_n \xrightarrow{C \otimes_R d_n^A} C \otimes_R B_{n-1}(A) \longrightarrow 0.$$

So  $Z_n(C \otimes_R P) = C \otimes_R Z_n(P) \in \mathcal{B}_C^0(R)$  and  $B_{n-1}(C \otimes_R P) = C \otimes_R B_{n-1}(P) \in \mathcal{B}_C^0(R)$ . Similarly, consider the exact sequence  $0 \rightarrow B_n(A) \rightarrow Z_n(A) \rightarrow H_n(A) \rightarrow 0$ , and we have, from the exact sequence

$$0 \longrightarrow C \otimes_R B_n(A) \longrightarrow C \otimes_R Z_n(A) \longrightarrow C \otimes_R H_n(A) \longrightarrow 0,$$

that  $H_n(C \otimes_R A) = C \otimes_R H_n(A) \in \mathcal{B}_C^0(R)$ .

The proof that  $\mathbf{R}\text{Hom}_R(C, -)$  takes  $\text{CE-}\mathcal{B}_C(R)$  into  $\text{CE-}\mathcal{A}_C(R)$  is similar. Finally, it follows from Proposition 4.2 that there are inclusions of categories  $\text{CE-}\mathcal{A}_C(R) \subseteq \mathcal{A}_C(R)$  and  $\text{CE-}\mathcal{B}_C(R) \subseteq \mathcal{B}_C(R)$ , and hence the equivalence of categories is immediate by [4, Theorem 4.6]. This completes the proof.  $\square$

The subcategory  $\{X \in \mathcal{D}_b(R) | X \simeq P \text{ with } P \text{ a bounded C-E projective complex}\}$  of  $\mathcal{D}_b(R)$  is denoted by  $\text{CE-}\overline{\mathcal{P}}(R)$ . Similarly,  $\text{CE-}\overline{\mathcal{P}}_C(R)$  denotes the subcategory of  $\mathcal{D}_b(R)$  consisting of complexes isomorphic to bounded C-E  $C$ -projective complexes.

By [14, Lemmas 4.1, 5.1], the Auslander class  $\mathcal{A}_C^0(R)$  contains every flat (projective)  $R$ -module and every  $C$ -injective  $R$ -module, and the Bass class  $\mathcal{B}_C^0(R)$  contains every injective  $R$ -module and every  $C$ -projective  $R$ -module. We have:

**Proposition 4.4.** (1) *There are inclusions of categories:*

$$\text{CE-}\overline{\mathcal{P}}(R) \subseteq \overline{\mathcal{P}}(R) \cap \text{CE-}\mathcal{A}_C(R), \text{CE-}\overline{\mathcal{P}_C}(R) \subseteq \overline{\mathcal{P}_C}(R) \cap \text{CE-}\mathcal{B}_C(R).$$

(2) *There is an equivalence of categories:*

$$\text{CE-}\overline{\mathcal{P}}(R) \xrightleftharpoons[\mathbf{R}\text{Hom}_R(C, -)]{C \otimes_{\mathbf{R}}^{\mathbf{L}} -} \text{CE-}\overline{\mathcal{P}_C}(R).$$

*Proof.* The inclusions of categories are obvious. We only need to prove (2).

Let  $X \in \text{CE-}\overline{\mathcal{P}}(R)$ . Then there exists an isomorphism  $X \simeq P$  in  $\mathcal{D}(R)$ , where  $P$  is a bounded C-E projective complex. By Corollary 2.4, we may choose  $P$  to be a graded module of projectives. Note that  $C \otimes_{\mathbf{R}}^{\mathbf{L}} X \simeq C \otimes_R P$  and  $C \otimes_R P$  is a graded module of with items being  $C$ -projective modules. Then  $C \otimes_{\mathbf{R}}^{\mathbf{L}} X$  is in  $\text{CE-}\overline{\mathcal{P}_C}(R)$ .

For any  $X \in \text{CE-}\overline{\mathcal{P}_C}(R)$ , it is proved similarly that  $\mathbf{R}\text{Hom}_R(C, X) \in \text{CE-}\overline{\mathcal{P}}(R)$ . Moreover, by Proposition 4.3 the equivalence of categories holds.  $\square$

We denote by  $\mathcal{G}(\mathcal{P})$ ,  $\mathcal{G}(\mathcal{I})$ ,  $\mathcal{G}(\mathcal{P}_C)$  and  $\mathcal{G}(\mathcal{I}_C)$  the class of Gorenstein projective, Gorenstein injective,  $C$ -Gorenstein projective and  $C$ -Gorenstein injective modules respectively.

Recall from [13] and [21] that a complete  $\mathcal{P}\mathcal{P}_C$ -resolution is a complex  $X$  of  $R$ -modules satisfying: (1) The complex  $X$  is exact and  $\text{Hom}_R(-, \mathcal{P}_C)$ -exact; (2) The  $R$ -module  $X_i$  is projective if  $i \geq 0$  and  $X_i$  is  $C$ -projective if  $i < 0$ . An  $R$ -module  $M$  is  $\mathcal{G}_C$ -projective if there exists a complete  $\mathcal{P}\mathcal{P}_C$ -resolution  $X$  such that  $M \cong Z_{-1}(X)$ . Dually, a complete  $\mathcal{I}_C\mathcal{I}$ -coresolution and a  $\mathcal{G}_C$ -injective module are defined. The classes of  $\mathcal{G}_C$ -projective and  $\mathcal{G}_C$ -injective modules are denoted by  $\mathcal{GP}_C$  and  $\mathcal{GI}_C$  respectively.

In the following lemma, (1) is from [17, Proposition 5.2] or [10, Proposition 3.6], and (2) is from [10, Theorem 3.11].

**Lemma 4.5.** (1)  $\mathcal{G}(\mathcal{P}_C) = \mathcal{GP}_C \cap \mathcal{B}_C^0(R)$  and  $\mathcal{G}(\mathcal{I}_C) = \mathcal{GI}_C \cap \mathcal{A}_C^0(R)$ .

(2) *Let  $M$  be an  $R$ -module. If  $M \in \mathcal{G}(\mathcal{P}) \cap \mathcal{A}_C^0(R)$ , then  $C \otimes_R M \in \mathcal{G}(\mathcal{P}_C)$ ; if  $M \in \mathcal{G}(\mathcal{P}_C)$ , then  $\text{Hom}_R(C, M) \in \mathcal{G}(\mathcal{P}) \cap \mathcal{A}_C^0(R)$ . Dually, if  $M \in \mathcal{G}(\mathcal{I}_C)$ , then  $C \otimes_R M \in \mathcal{G}(\mathcal{I}) \cap \mathcal{B}_C^0(R)$ ; if  $M \in \mathcal{G}(\mathcal{I}) \cap \mathcal{B}_C^0(R)$ , then  $\text{Hom}_R(C, M) \in \mathcal{G}(\mathcal{I}_C)$ .*

In the next, we consider the subcategory  $\{X \in \mathcal{D}_b(R) | X \simeq G\}$  of  $\mathcal{D}_b(R)$ . If  $G$  is a bounded C-E Gorenstein projective complex, then it is denoted by  $\text{CE-}\overline{\mathcal{G}(\mathcal{P})}(R)$ ; if  $G$  is a bounded C-E  $C$ -Gorenstein projective complex, then we use  $\text{CE-}\overline{\mathcal{G}(\mathcal{P}_C)}(R)$  to indicate it.

**Proposition 4.6.** (1) *There are inclusions of categories:*

$$\text{CE-}\overline{\mathcal{P}}(R) \subseteq \text{CE-}\overline{\mathcal{G}(\mathcal{P})}(R) \cap \text{CE-}\mathcal{A}_C(R), \text{CE-}\overline{\mathcal{P}_C}(R) \subseteq \text{CE-}\overline{\mathcal{G}(\mathcal{P}_C)}(R) \subseteq \text{CE-}\mathcal{B}_C(R).$$

(2) *There is an equivalence of categories:*

$$\text{CE-}\overline{\mathcal{G}(\mathcal{P})}(R) \cap \text{CE-}\mathcal{A}_C(R) \xrightleftharpoons[\mathbf{R}\text{Hom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} \text{CE-}\overline{\mathcal{G}(\mathcal{P}_C)}(R).$$

*Proof.* Let  $X$  be a complex in  $\mathcal{D}_b(R)$ . If  $X \in \text{CE-}\overline{\mathcal{G}(\mathcal{P})}(R) \cap \text{CE-}\mathcal{A}_C(R)$ , then  $X \simeq G$ , where  $G$  is a bounded C-E Gorenstein projective complex and is in  $\text{CE-}\mathcal{A}_C(R)$ . By the argument in Proposition 4.2,  $C \otimes_R^{\mathbf{L}} X \simeq C \otimes_R G$ .

For  $X \in \text{CE-}\overline{\mathcal{G}(\mathcal{P}_C)}(R)$ , it is proved similarly that  $X \simeq G$  where  $G$  is a bounded C-E  $C$ -Gorenstein projective complex, and moreover,  $\mathbf{R}\text{Hom}_R(C, X) \simeq \text{Hom}_R(C, G)$ . Then, the assertions follow by Lemma 4.5 and Proposition 4.3.  $\square$

**Remark 4.7.** *The dual results hold, and then we have the following diagram:*

$$\begin{array}{ccc} \mathcal{A}_C(R) & \xrightleftharpoons[\mathbf{R}\text{Hom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \mathcal{B}_C(R) \\ \uparrow & & \uparrow \\ \text{CE-}\mathcal{A}_C(R) & \xrightleftharpoons[\mathbf{R}\text{Hom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \text{CE-}\mathcal{B}_C(R) \\ \uparrow & & \uparrow \\ \text{CE-}\overline{\mathcal{G}(\mathcal{I}_C)}(R) & \xrightleftharpoons[\mathbf{R}\text{Hom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \text{CE-}\overline{\mathcal{G}(\mathcal{I})}(R) \cap \text{CE-}\mathcal{B}_C(R) \\ \uparrow & & \uparrow \\ \text{CE-}\overline{\mathcal{I}_C}(R) & \xrightleftharpoons[\mathbf{R}\text{Hom}_R(C, -)]{C \otimes_R^{\mathbf{L}} -} & \text{CE-}\overline{\mathcal{I}}(R) \end{array}$$

## 5. Cartan-Eilenberg homological dimensions

In this section, we are devoted to study homological dimensions of complexes with respect to C-E complexes. The symbol  $\mathcal{D}_{\sqsupset}(R)$  (resp.  $\mathcal{D}_{\sqsubset}(R)$ ) stands for the full subcategory of  $\mathcal{D}(R)$  consisting of homologically right-bounded (resp. homologically left-bounded) complexes.

**Definition 5.1.** *Let  $X$  be a complex in  $\mathcal{D}_{\sqsupset}(R)$ . Consider the invariant*

$$\inf\{\sup\{i \in \mathbb{Z} \mid P_i \neq 0\} \mid X \simeq P \text{ with } P \in \mathcal{C}_{\mathcal{X}}(R)\};$$

*if  $\mathcal{C}_{\mathcal{X}}(R)$  is the category of C-E projective complexes, then it is said to be C-E projective dimension of  $X$ , and is denoted by  $\text{CE-pd}_R X$ ; if  $\mathcal{C}_{\mathcal{X}}(R)$  is the category of C-E Gorenstein projective complexes, then it is said to be C-E Gorenstein projective dimension of  $X$ , and is denoted by  $\text{CE-Gpd}_R X$ .*

Next, we will show that under reasonable conditions, the set taken in the above definition is not empty.

Recall that a complex  $P$  is DG-projective provided that  $P_n$  is a projective module for any  $n \in \mathbb{Z}$  such that every chain map from  $P$  to an exact complex  $E$  is homotopic to zero; which is also termed semi-projective complex. A complex is projective if and only if it is both exact and DG-projective. By [6, section 10], every projective complex is C-E projective and every C-E projective complex is DG-projective.

**Lemma 5.2.** *Let  $X$  be a DG-projective complex with  $H(X)$  consisting of projective modules. Then there exists a split exact sequence  $0 \rightarrow K \rightarrow P \rightarrow X \rightarrow 0$ , where  $P$  is C-E projective and  $K$  is projective. Consequently, a complex  $P$  is C-E projective if and only if  $P$  is DG-projective with  $H(P)$  a complex of projective modules.*

*Proof.* For any  $n \in \mathbb{Z}$ , consider the exact sequence  $0 \rightarrow L_n \rightarrow Q_n \rightarrow B_n(X) \rightarrow 0$  with  $Q_n$  projective. By using the Horseshoe Lemma, we have the following commutative diagrams:

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & L_n & \longrightarrow & L_n & \longrightarrow & 0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q_n & \longrightarrow & Q_n \oplus H_n(X) & \longrightarrow & H_n(X) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B_n(X) & \longrightarrow & Z_n(X) & \longrightarrow & H_n(X) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

and

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & L_n & \longrightarrow & K_n & \longrightarrow & L_{n-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q_n \oplus H_n(X) & \longrightarrow & Q_n \oplus H_n(X) \oplus Q_{n-1} & \longrightarrow & Q_{n-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z_n(X) & \longrightarrow & X_n & \longrightarrow & B_{n-1}(X) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

Let  $P_n = Q_n \oplus H_n(X) \oplus Q_{n-1}$ . Then  $P = (P_n, d_n)$  is a complex with the differential  $d_n : P_n \rightarrow P_{n-1}$  which is given by  $d_n(x, y, z) = (z, 0, 0)$ . It is direct to check that  $P$  is a C-E projective complex. Moreover, we have an exact complex  $K$  by pasting the exact sequences  $0 \rightarrow L_n \rightarrow K_n \rightarrow L_{n-1} \rightarrow 0$  together;  $K$  is also DG-projective since both  $X$  and  $P$  are so, and then  $K$  is projective. Note that the exact sequence  $0 \rightarrow K \rightarrow P \rightarrow X \rightarrow 0$  is split degreewise and  $K$  is contractible, hence it is split. This implies that  $X$  is a direct summand of  $P$ , and then is DG-projective.  $\square$

**Proposition 5.3.** *Let  $X$  be a complex. Then  $X \simeq P$  for a C-E projective complex  $P$  if and only if  $H(X)$  is a complex of projective modules. In this case, one can chose  $P$  to be DG-projective.*

*Proof.* It is well known that for any complex  $X$ , there exist a (epic) quasi-isomorphism  $P \rightarrow X$  with  $P$  DG-projective. Then the assertion follows immediately by Lemma 5.2.  $\square$

**Remark 5.4.** (1) *For any complex  $X \in \mathcal{D}_{\square}(R)$ , it follows from the definition that  $\sup(X) \leq \text{CE-Gpd}_R X \leq \text{CE-pd}_R X$ . If  $X \simeq 0$ , i.e.  $X$  is exact, then  $\text{CE-pd}_R X = -\infty = \text{CE-Gpd}_R X$ .*

(2) *Recall that when  $\mathcal{C}_X(R)$  in the above definition is the category of right-bounded complexes of projective modules, then it is projective dimension of  $X$ , denoted by  $\text{pd}_R X$ ; when  $\mathcal{C}_X(R)$  is the category of right-bounded complexes of Gorenstein projective modules, then it is Gorenstein projective dimension in the sense of Christensen et. al [3, 5], and is denoted by  $\text{Gpd}_R X$ . Moreover, we have  $\text{pd}_R X \leq \text{CE-pd}_R X$ ,  $\text{Gpd}_R X \leq \text{CE-Gpd}_R X$ .*

**Theorem 5.5.** *Let  $X \in \mathcal{D}_{\square}(R)$ . Assume that  $H(X)$  is a complex of projective modules. Then the following are equivalent:*

- (1)  *$\text{CE-pd}_R X$  is finite.*
- (2)  *$\text{pd}_R X$ ,  $\text{pd}_R Z(X)$ ,  $\text{pd}_R B(X)$  and  $\text{pd}_R H(X)$  are all finite.*

*Proof.* (1) $\implies$ (2) Since  $\text{CE-pd}_R X < \infty$ , it is trivial that  $\text{pd}_R X < \infty$ . Noting that  $X \simeq P$  for a bounded C-E projective complex  $P$ ,  $H(X)$  is isomorphic to the bounded complex of projective modules  $H(P)$ , and then  $\text{pd}_R H(X) < \infty$ .

By [19, 1.4.3], in an exact sequence of complexes, if two of them have finite projective dimension, so does the third. Consider exact sequences  $0 \rightarrow B(X) \rightarrow Z(X) \rightarrow H(X) \rightarrow 0$  and  $0 \rightarrow Z(X) \rightarrow X \rightarrow B(X)[1] \rightarrow 0$ , where  $B(X)[1]$  is the complex obtained by shifting  $B(X)$  one-degree to the left. If either  $\text{pd}_R Z(X) = \infty$  or  $\text{pd}_R B(X) = \infty$ , a contradiction will occur. It yields that both  $\text{pd}_R Z(X)$  and  $\text{pd}_R B(X)$  are finite.



(2) $\implies$ (1) We set  $s = \text{pd}_R X$  and  $i = \inf(X)$ . Then  $X \simeq P \simeq P_{(s,i)}$ , where  $P$  is C-E projective and

$$P_{(s,i)} = 0 \longrightarrow P_s/B_s(P) \longrightarrow P_{s-1} \longrightarrow \cdots \longrightarrow P_{i+1} \longrightarrow Z_i(P) \longrightarrow 0$$

is the truncated complex of  $P$ . By [1, Theorem 2.4.P.],  $P_s/B_s(P)$  is a projective module, and then  $P_{(s,i)}$  is also a C-E projective complex. Hence  $\text{CE-pd}_R X \leq s$ .  $\square$

**Corollary 5.6.** *Let  $X \in \mathcal{D}_\square(R)$  be a complex of finite C-E projective dimension. Then  $\text{CE-pd}_R X = \text{pd}_R X$ .*

Modules with excellent duality properties have turned out to be a powerful tool [12]. Recall that a semi-dualizing module  $C$  is dualizing provided that  $C$  has finite injective dimension. For C-E Gorenstein projective dimension, we have the following results from Theorem C in introduction.

**Theorem 5.7.** *Let  $X \in \mathcal{D}_\square(R)$ . Assume that  $X \simeq G$  for a C-E Gorenstein projective complex  $G$ . Then the following are equivalent:*

- (1)  $\text{CE-Gpd}_R X$  is finite.
- (2)  $\text{Gpd}_R X$ ,  $\text{Gpd}_R Z(X)$ ,  $\text{Gpd}_R B(X)$  and  $\text{Gpd}_R H(X)$  are all finite.

*Moreover, if  $R$  is a noetherian ring with a dualizing module  $C$ , then the above are equivalent to:*

- (3)  $X \in \text{CE-}\mathcal{A}_C(R)$ .
- (4)  $X$ ,  $Z(X)$ ,  $B(X)$  and  $H(X)$  are in  $\mathcal{A}_C(R)$ .

*Proof.* By [19, Theorem 3.9 (1)], in an exact sequence of complexes, if two of them have finite Gorenstein projective dimension, then so does the third. Then (1) $\implies$ (2) follows by an argument analogous to the proof in Theorem 5.5. For (2) $\implies$ (1), we set  $s = \text{Gpd}_R X$ ,  $i = \inf(X)$  and assume that  $X \simeq G$  for a C-E Gorenstein projective complex  $G$ . By [5, Theorem 3.1],  $G_s/B_s(G)$  is a Gorenstein projective module. Then  $X \simeq G_{(s,i)}$  with  $G_{(s,i)}$  a C-E Gorenstein projective complex. Hence  $\text{CE-Gpd}_R X \leq s$ .

Now let  $R$  be a noetherian ring with a dualizing module  $C$ . It follows from [7, Proposition 3.9] that every Gorenstein projective module is in  $\mathcal{A}_C^0(R)$ . Hence the condition  $\text{CE-Gpd}_R X < \infty$  implies  $X \simeq G$  for a bounded C-E Gorenstein projective complex  $G$ , and hence  $X \simeq G \in \text{CE-}\mathcal{A}_C(R)$ . Then (1) $\implies$ (3) follows. It is immediate from Proposition 4.2 that (3) $\implies$ (4), and (2) $\iff$ (4) follows by [5, Theorem 4.1].  $\square$

**Corollary 5.8.** *Let  $X \in \mathcal{D}_\square(R)$  be a complex of finite C-E Gorenstein projective dimension. Then  $\text{CE-Gpd}_R X = \text{Gpd}_R X$ .*

Dually, for a homologically left-bounded complex  $X$ , one can define C-E injective dimension, denoted by  $\text{CE-id}_R X$ , to be  $\inf\{\sup\{i \in \mathbb{Z} \mid I_{-i} \neq 0\} \mid X \simeq$

$I$  with  $I$  C-E injective}. Similarly, C-E Gorenstein injective dimension of  $X$  is defined with  $\inf\{\sup\{i \in \mathbb{Z} \mid E_{-i} \neq 0\} \mid X \simeq E \text{ with } E \text{ C-E Gorenstein injective}\}$ , and is denoted by  $\text{CE-Gid}_R X$ .

**Theorem 5.9.** *Let  $X \in \mathcal{D}_{\square}(R)$ . Assume that  $X \simeq I$  for a C-E injective complex  $I$  (equivalently,  $H(X)$  is a complex of injective modules). Then the following are equivalent:*

- (1)  $\text{CE-id}_R X$  is finite.
- (2)  $\text{id}_R X$ ,  $\text{id}_R Z(X)$ ,  $\text{id}_R B(X)$  and  $\text{id}_R H(X)$  are all finite.

**Corollary 5.10.** *Let  $X \in \mathcal{D}_{\square}(R)$  be a complex of finite C-E injective dimension. Then  $\text{CE-id}_R X = \text{id}_R X$ .*

**Theorem 5.11.** *Let  $X \in \mathcal{D}_{\square}(R)$ . Assume that  $X \simeq E$  for a C-E Gorenstein injective complex  $E$ . Then the following are equivalent:*

- (1)  $\text{CE-Gid}_R X$  is finite.
- (2)  $\text{Gid}_R X$ ,  $\text{Gid}_R Z(X)$ ,  $\text{Gid}_R B(X)$  and  $\text{Gid}_R H(X)$  are all finite.

*Moreover, if  $R$  is a noetherian ring with a dualizing module  $C$ , then the above are equivalent to:*

- (3)  $X \in \text{CE-}\mathcal{B}_C(R)$ .
- (4)  $X$ ,  $Z(X)$ ,  $B(X)$  and  $H(X)$  are in  $\mathcal{B}_C(R)$ .

**Corollary 5.12.** *Let  $X \in \mathcal{D}_{\square}(R)$  be a complex of finite C-E Gorenstein injective dimension. Then  $\text{CE-Gid}_R X = \text{Gid}_R X$ .*

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